Math 3210

Tutorial 3

Example 0: Review from last tutorial

Defination of a hyper plane in n dimension

method one:

$n \cdot X = C$

method two:

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = C$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = C$$

If a set of points gives the same solution to a linear problem, then any convex combination of the points give the same solution.

proved in last tutorial:

If a collection of points in $\mathbb{R}^n$

$S_1 = \begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix} \cdots S_m = \begin{pmatrix} x_1^m \\ \vdots \\ x_n^m \end{pmatrix}$ is on the same hyper plane.

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = C$

Then their convex combination must be on the Hyperplane as well.

Then if the set of points $S_1 \cdots S_m$ gives the same solution when applied to the linear system.

$$Z = C_1^1 X_1^1 + C_2^2 X_2^2 + C_3^3 X_3^3 + \cdots + C_n X_n = \cancel{C}$$

i.e

$$\left.\begin{array}{l} C_1 X_1^1 + \cdots + C_n X_n^1 = C \\ \vdots \\ C_1 X_1^m + \cdots + C_n X_n^m = C \end{array}\right\}$$

S all on the same hyper plane.

Some big picture:

Consider the set by adding surplus variable.

$$\underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}}_{S} \to \underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ S_1 \\ \vdots \\ S_m \end{pmatrix}}_{S'} \text{ in space.}$$

in terms of matrix

Constrain

$$A_1^1 X_1 + A_2 X_2 + \cdots + A_n^1 X_n \leq b_1 \qquad A_1 X_1 + \cdots + A_n^1 X_n + S_1 = b_1$$

$$A_1^m X_1 + A_2 X_2 \cdots + A_n^m X_n \leq b_n \qquad A_1^m X_1 + \cdots + A_n^m X_n + S_m = b_n$$

$$X_1 \cdots X_n \geq 0 \qquad X_1 \cdots X_n, S_1, \cdots, S_m \geq 0$$

$$\underbrace{AX \leq b}_{S} \qquad \underbrace{A'X = b}_{S'}$$

$$m\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ & & \\ a_{m1} & & a_{mn} \end{pmatrix}_{n}\begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix} \qquad m\underbrace{\left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{n1} & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \\ a_{m1} & & a_{mn} & & & 1 \end{array}\right)}_{n+m}\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_m \end{pmatrix}$$

$$\circledast \quad = \begin{pmatrix} a_{11}x_1 + \cdots a_{n1}x_n \\ \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \qquad \begin{pmatrix} a_{11}x_1 + \cdots + a_{m1}x_n + s_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + s_m \end{pmatrix}$$

Example 2: Prove that if P' is not an extreme point on S', then P is not an extreme point on S.

$\exists P_1', P_2' \subset S'$ s.t. $P' = \lambda P_1' + (1-\lambda) P_2'$ where $0 \le \lambda \le 1$

$$P_1' = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n^1 \\ s_1^1 \\ \vdots \\ s_m^1 \end{pmatrix} \qquad P_2' = \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \\ s_1^2 \\ \vdots \\ s_m^2 \end{pmatrix} \qquad P' = \begin{pmatrix} x_1^P \\ \vdots \\ x_n^P \\ s_1^P \\ \vdots \\ s_m^P \end{pmatrix} = \begin{pmatrix} P \\ s_1^P \\ \vdots \\ s_m^P \end{pmatrix}$$

take $P_1 = \begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix}$ $P_2 = \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}$ $\lambda P_1 + (1-\lambda) P_2 = P$.

$\circledast$

for P_1, $$\underbrace{AX = \begin{pmatrix} a_{11}x_1^1 + a_{12}x_2^1 + \cdots + a_{1n}x_n^1 \\ \\ a_{m1}x_1^1 + \cdots + a_{mn}x_n^1 \end{pmatrix}}_{\text{must belong to } S} \le \underbrace{\begin{pmatrix} a_{11}x_1^1 + \cdots + a_{1n}x_n^1 + s_1 \\ \\ \\ \end{pmatrix} = b}_{\text{since } P_1' \text{ belong to } S'}$$

Different from:

Basic LLP problem

$$\begin{array}{ll} \text{maxormin} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n(\leq,\ =,\ \geq)b_1, \\ \quad\vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n(\leq,\ =,\ \geq)b_m, \end{cases} \end{array}$$

All a,x,b assume to be real

Canonical From

$$\begin{array}{ll} \text{maxormin} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1, \\ \quad\vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m, \end{cases} \\ \text{where} & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{array}$$

Note that if all the Bs are non negative, then it is called a Feasible Canonical from.

Standard from:

$$\begin{array}{ll} \max & z = c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & \begin{cases} a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \quad i = 1, 2, \cdots, m \\ \qquad\qquad\qquad x_j \geq 0, \quad j = 1, 2, \cdots, n\ . \end{cases} \end{array}$$

Example 3: Dealing with free variables:

Example: Max: $Z = 2x_1 + 5x_2 + 3x_3 - 15x_4 + 10x_5$

Subject to: $2x_1 + x_2 + 2x_3 + x_4 = 80$

$x_1 + x_2 + 2x_3 + x_5 = 65$

$x_1, x_2, x_3 \geq 0$

x_4, x_5 free.

$x^4 = u^4 - v^4$

$x^5 = u^5 - v^5$

Max $Z = 2x_1 + 5x_2 + 3x_3 - 15x^4 + 15v^4 + 10x^5 - 10v^5$

Subject to: $2x_1 + x_2 + 2x_3 + u^4 - v^4 = 80$

$x_1 + x_2 + 2x_3 + u^5 - v^5 = 65$

$x_1, x_2, x_3, v_4, v_5, u_4, u_5 \geq 0$ //

Example 4: Making right hand side positive

Making the right hand side positive.

$-x_1 + x_2 \leq -3$

$x_1 - x_2 \geq 3$

$\cancel{x_1 - x_2 + s_1}$ $x_1 - x_2 - s_1 = 3$

$s_1 \geq 0$.

Example 5: Changing constrain:

Changing constrain: convert to canonical.

$\max: Z = X_1 + 2X_2 + X_3$

subject to $X_1 + 2X_2 + X_3 = 1$

$X_1 + X_2 \geq 4$

$X_1, X_2 \geq 0$

$X_3 \leq 3$

$X_1 + 2X_2 + X_3 = 1$

$= X_1 + 2X_2 + X_3 - S_1 \leq 1$

$-(X_1 + X_2) \leq -4$

$\underbrace{3 - X_3}_{u} \geq 0$

$\max \ 2X_1 + 2X_2 + (3 - u) = Z$

$\Rightarrow \max \ 2X_1 + 2X_2 - u$

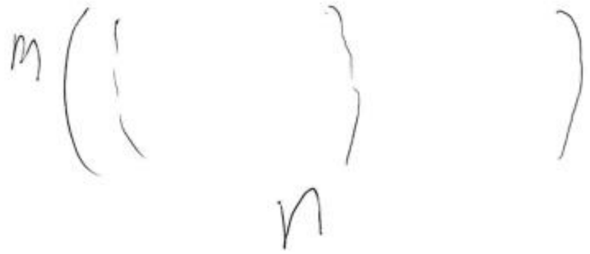
subject to : $X_1 + 2X_2 + (3 - u) - S_1 \leq 1$

$-(X_1 + X_2) \leq -4$

$X_1, X_2, u, \geq 0.$

A simple review on basic solution and its usage:

each set of Basic variables give us an $m \times m$ matrix B.



each basic solution give us a corner/ extreme points → one/ some of the points give us the optimal solution.

Example 6: How to speed up computation

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 8 \end{bmatrix} \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \quad b = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

$$\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}$$

$$\max \quad 2x_1 + 5x_2 + 6x_3 + 8x_4 = z$$

$$\text{W.R.T} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ x_1 + x_4 &= 2 \end{aligned}$$

Chose a basic solution.

$B = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ $\quad$ $\cancel{Z = \underbrace{\begin{bmatrix} 1 & 4 \\ & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 2 \\ 8 \end{bmatrix}}_{C} =}$

$\therefore Bx = b \quad x_{1,4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C_{1,4} \cdot \bar{X}_{1,4} = 10 = Z_{1,4}$

note $C_{1,4} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

$B^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$

Compute y : $y_1 = B^{-1} A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y_2 = B^{-1} A_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$, $y_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $y_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Compute Z of the non basic

$Z_2 = C_{1,4} \cdot y_2 = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = 4$

$Z_3 = C_{1,4} \cdot y_3 = \quad = 6.$

Compute C-Z, only if it is positive does we worth putting the vector ~~of~~ variable as basic.

$\boxed{C_2 - Z_2 = 1} \rightarrow$ try $\boxed{x_2}$ as basic.

$C_3 - Z_3 = 0$

look at y_2

$\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \to$ the second term non negative,
i.e kick away the second vector

this always work since
B is a square matrix.